

– 1 Krall-Jacobi polynomials

Luc Vinet[†], Guo-Fu Yu^{‡,†} and Alexei Zhedanov[§]

[†] Centre de Recherches Mathématiques, Université de Montréal,
C.P.6128, Centre-ville Station, Montréal, Québec, H3C 3J7, Canada

[‡]Department of Mathematics, Shanghai Jiao Tong University,
Shanghai 200240, P.R. China

[§] Donetsk Institute for Physics and Technology. Donetsk 83114, Ukraine

Abstract

We study a family of orthogonal polynomials which satisfy (apart from a 3-term recurrence relation) an eigenvalue equation involving a third order differential operator of Dunkl-type. These polynomials can be obtained from a Geronimus transformation of the little q -Jacobi polynomials in the limit $q = -1$.

Keywords: Jacobi polynomials, little q -Jacobi polynomials, Geronimus transformation.

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1 Introduction

Significant advances have been realized in the characterization of recently discovered families of -1 orthogonal polynomials (OPs). The striking feature of these OPs is that they are classical or bispectral and that they satisfy eigenvalue equations involving Dunkl-type operators [1] in addition to the mandatory 3-term recurrence relation. They have arisen already in a number of physical problems [2]-[7] and are connected with Jordan algebras [8].

At the top of the emerging -1 scheme are the Bannai-Ito polynomials and their kernel partners, the complementary Bannai-Ito polynomials [9]. Both sets depend on 4 parameters. The Bannai-Ito polynomials are the eigensolutions of the most general operator which is of first order in Dunkl shifts, (i.e. first order in the operators T and R defined by $Tf(x) = f(x+1)$, $Rf(x) = f(-x)$ on a function $f(x)$) and which stabilizes polynomials of given degrees. The Bannai-Ito polynomials are positive definite and orthogonal on a finite set of $N + 1$ points. In the limits where $N \rightarrow \infty$, they tend to the big -1 Jacobi polynomials [10] which are orthogonal on $[-1, -c] \cup [c, 1]$. When $c = 0$, the little -1 Jacobi polynomials [11] arise as a special case. A Bochner-type theorem establishes [12] that the big and little -1 polynomials are the only families of OPs satisfying a differential-difference eigenvalue equation which is of first order in Dunkl-type operators.

A significant property that justifies the nomenclature is that these -1 orthogonal polynomials can be obtained from appropriate $q \rightarrow -1$ limits of q - polynomials.

The Bannai-Ito polynomials and their descendants all possess the Leonard duality property. (In fact, this led to their initial identification [13].) The dual -1 Hahn polynomials [14] together with the generalized Gegenbauer [15] and Hermite polynomials [16, 17] are also bispectral but, obeying eigenvalue equations of second order in Dunkl operators, they fall beyond the scope of Leonard duality.

We continue here the exploration of -1 orthogonal polynomials in that vein and look for another class of -1 OPs verifying a higher differential-difference equation.

In the wake of Krall's classification of OPs satisfying a fourth order differential equations [18], it is appreciated that the addition of discrete masses to the measure leads to OPs verifying higher order equations [19, 20, 21].

In this connection, we present here a generalization of the little -1 Jacobi polynomials with the following features: these orthogonal polynomials obey a differential-difference equation of third order in Dunkl operators and a mass is located at the middle of the orthogonality interval.

The outline of the paper is the following. In section 2, we offer a brief review of useful results on little q -Jacobi polynomials. In section 3, we introduce the generalized little q -Jacobi polynomials [22] that are eigensolutions of higher order q -difference equations. In section 4, focusing for definiteness on one of the

simpler cases, we obtain and characterize a set of generalized little -1 Jacobi polynomials by taking an appropriate $q \rightarrow -1$ limit of certain polynomials of the proceeding section. The paper ends with concluding remarks.

2 Little q -Jacobi polynomials

The monic little q -Jacobi polynomials are defined as

$$P_n(x; a, b) = (-1)^n \frac{q^{n(n-1)/2} (aq; q)_n}{(abq^{n+1}; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| qx \right) \quad (2.1)$$

where $(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$ is the q -shifted factorial and ${}_2\phi_1$ denotes the q -hypergeometric function.

The orthogonality relation is

$$\sum_{k=0}^{\infty} w_k P_n(q^k; a, b) P_m(q^k; a, b) = h_n \delta_{nm}, \quad (2.2)$$

where h_n are appropriate normalization constants, and the normalized weight function is

$$w_k = \frac{(aq; q)_{\infty}}{(abq^2; q)_{\infty}} \frac{(bq; q)_k (aq)^k}{(q; q)_k}. \quad (2.3)$$

It is assumed that $0 < aq < 1, b < q^{-1}$. The expansion coefficients of the little q -Jacobi polynomials in $P_n(x; a, b) = \sum_{s=0}^n B_n^{(s)} x^{n-s}$ are

$$B_n^{(s)} = b^{-s} \frac{(q^{-n}, a^{-1}q^{-n}; q)_s}{(q, a^{-1}b^{-1}q^{-2n}; q)_s}. \quad (2.4)$$

It is known that the little q -Jacobi polynomials satisfy a second-order difference equation [23].

Introduce the functions of the second kind,

$$Q_n(z) = \int_a^b \frac{P_n(x)w(x)}{z-x} dx, \quad (2.5)$$

where $w(x)$ is assumed to be the normalized weight function, i.e.

$$\int_a^b w(x) dx = 1.$$

The values of the functions $Q_n(z)$, at $z = 0$ (an accumulation point of the orthogonality measure), are :

$$Q_n(0; a, b) = - \sum_{k=0}^{\infty} \frac{P_n(q^k; a, b) w_k}{q^k}. \quad (2.6)$$

Using the q -binomial theorem and the q -Saalschütz formula (see, e.g., [24]) we have

$$Q_n(0; a, b) = (-1)^{n+1} a^n q^{n(n-1)/2} \frac{1-abq}{1-a} \frac{(q; q)_n (bq; q)_n}{(abq; q)_n (abq^{n+1}; q)_n}. \quad (2.7)$$

We note that if $a = q^j, j = 1, 2, 3, \dots$, then

$$\begin{aligned} \Phi_n &= Q_n(0; q^j, b) + M P_n(0; q^j, b) \\ &= (-1)^n q^{n(n-1)/2} \frac{(q^{j+1}; q)_n}{(bq^{n+j+1}; q)_n} \left(M - q^{nj} \frac{(1-bq^{j+1})(bq; q)_j (q; q)_j}{(1-q^j)(q^{n+1}; q)_j (bq^{n+1}; q)_j} \right). \end{aligned} \quad (2.8)$$

3 Transformed q -Jacobi polynomials

Let $P_n(x)$ be orthogonal polynomials with measure localized on the interval $[a, b]$. Let $w(x)$ be the corresponding normalized unit weight function and $Q_n(z)$ be defined by (2.5). Let finally c be a point beyond the orthogonality interval $[a, b]$ such that $Q_n(c)$ exists.

Consider the Geronimus transformation [25, 26] of the polynomials $P_n(x)$ at the point $x = c$

$$\tilde{P}_n(x) = \mathcal{G}P_n(x) = P_n(x) - \frac{\Phi_n}{\Phi_{n-1}}P_{n-1}(x), n = 1, 2, \dots, \tilde{P}_0(x) = 1, \quad (3.1)$$

where

$$\Phi_n = Q_n(c) + MP_n(c). \quad (3.2)$$

The weight function $\tilde{w}(x)$ of the polynomials $\mathcal{G}\{P_n(x)\}$ is

$$\tilde{w}(x) = \kappa \left(\frac{w(x)}{x - c} - M\delta(x - c) \right), \quad (3.3)$$

where κ is an appropriate normalization constant. The Geronimus transformation thus inserts a concentrated mass at the point $x = c$. The value of this mass depends on the parameter M . Now take for $P_n(x)$ the little q -Jacobi polynomials with $a = q^j, j = 1, 2, 3, \dots$, and perform the Geronimus transformation (3.1) with Φ_n given by (2.8). (In this case $c = 0$.)

The weight function $\tilde{w}(x)$ for the polynomials $\mathcal{G}(c)\{P_n(x)\}$ is

$$\tilde{w}(x) = \kappa \left(\sum_{k=0}^{\infty} \tilde{w}_k \delta(x - q^k) - M\delta(x) \right), \quad (3.4)$$

where

$$\tilde{w}_k = \frac{(q^{j+1}; q)_{\infty}}{(bq^{j+2}; q)_{\infty}} \frac{(bq^k; q)_k q^{jk}}{(q; q)_k}. \quad (3.5)$$

The coefficients $B_n^{(s)}$ in the expansion

$$\tilde{P}_n(x) = \mathcal{G}(0)\{P_n(x; q^j, b)\} = \sum_{s=0}^n B_n^{(s)} x^{n-s} \quad (3.6)$$

have been given in [22]. Let us introduce the operators

$$\mathcal{L}_q = \sum_{k=0}^{2N} a_k (q^{-N}x)^{T^{-N}} \mathcal{D}_q^k, \quad (3.7)$$

where

$$a_k(x) = \sum_{s=0}^k \alpha_{ks} x^s, \quad k = 0, 1, \dots, 2N. \quad (3.8)$$

T is the q -shift operator and \mathcal{D}_q the q -derivative operator. The operator \mathcal{L}_q is seen to be very practical in searching for orthogonal polynomials $P_n(x)$ satisfying eigenvalue equations of the kind

$$\mathcal{L}_q P_n(x) = \lambda_n P_n(x). \quad (3.9)$$

Consider the action of the operator \mathcal{L}_q upon the monomials x^n . From (3.7) and (3.8) we get

$$\mathcal{L}_q x^n = \sum_{s=0} A_n^{(s)} x^{n-s}, \quad (3.10)$$

where

$$A_n^{(s)} = q^{N(s-n)} [n][n-1] \cdots [n-s+1] \pi_s(q^n), \quad (3.11)$$

and

$$\pi_s(q^n) = \alpha_{s0} + \sum_{i=1}^{2N-s} \alpha_{s+i,i} [n-s][n-s-1] \cdots [n-s-i+1] \quad (3.12)$$

are polynomials in $z = q^n$ of degree not exceeding $2N - s$. It is clear that

$$A_n^{(s)} = 0, \quad s > 2N \quad (3.13)$$

(see [22]). The coefficients $A_n^{(s)}$ completely characterize the operator \mathcal{L}_q and $A_n^{(s)}$ are called the representation coefficients of the operator \mathcal{L}_q .

The coefficients $A_n^{(s)}$ for the q -difference operator \mathcal{L}_q that has the polynomials $\tilde{P}_n(x)$ (3.6) as eigenfunctions have been constructed in [22], they are the following

$$A_n^{(0)} = \lambda_n = \frac{M(q-1)q^{-n(j+1)-1}(q^n; q)_{j+1}(bq^n; q)_{j+1}}{1 - q^{-j-1}} - (q^{-n} - 1)(1 - bq^{n+j})(bq; q)_{j+1}(q; q)_{j-1}, \quad (3.14)$$

$$A_n^{(1)} = (1 - q^{-n}) \left(Mq^{j(1-n)}(q^{n+1}; q)_j(bq^n; q)_j(1 - q^{n-1}) - (q; q)_{j-1}(bq; q)_{j+1}(1 - q^{n+j-1}) \right) \quad (3.15)$$

$$A_n^{(2)} = M(q-1)q^{(2-n)(j+1)-1} \frac{(1 - q^{-j})(q^{n-2}; q)_{j+3}(bq^n; q)_{j-1}}{(q; q)_2}, \quad (3.16)$$

$$A_n^{(s)} = 0, \quad \text{if } s \geq j+2. \quad (3.17)$$

4 Limit of the Krall-Jacobi polynomials as $q \rightarrow -1$

In this section we construct the $q \rightarrow -1$ limit of the coefficients $A_n^{(s)}$. We take $j = 2$ and put

$$q = -e^\epsilon, \quad b = -e^{\beta\epsilon}, \quad (4.1)$$

Substituting into the $A_n^{(s)}$ as given in (3.14)-(3.17) and taking the limit $\epsilon \rightarrow 0$, we have

$$\frac{A_n^{(0)}}{\epsilon^3} \rightarrow \begin{cases} -8Mn(n+2)(n+1+\beta) + 8n(\beta+1)(\beta+3) & n \text{ even} \\ 8M(n+1)(n+\beta)(n+2+\beta) - 8(n+2+\beta)(\beta+1)(\beta+3) & n \text{ odd} \end{cases} \quad (4.2)$$

$$\frac{A_n^{(1)}}{\epsilon^3} \rightarrow \begin{cases} 8Mn(n+2)(n+1+\beta) - 8n(\beta+1)(\beta+3), & n \text{ even} \\ 8(\beta+1)(\beta+3)(n+1) - 8M(n^2-1)(n+\beta), & n \text{ odd} \end{cases} \quad (4.3)$$

$$\frac{A_n^{(2)}}{\epsilon^3} \rightarrow \begin{cases} 8Mn(n+2)(n-2), & n \text{ even} \\ -8M(n+1)(n-1)(n+\beta), & n \text{ odd} \end{cases} \quad (4.4)$$

$$\frac{A_n^{(3)}}{\epsilon^3} \rightarrow \begin{cases} -8Mn(n+2)(n-2), & n \text{ even} \\ 8M(n+1)(n-1)(n-3), & n \text{ odd} \end{cases}. \quad (4.5)$$

Consider the form of the q -difference equation (3.9) in this limit. We divide both sides of (3.9) by ϵ^3 and introduce the operator L_ϵ which acts on the polynomials $\tilde{P}_n(x)$ as

$$L_\epsilon \tilde{P}_n(x) = \epsilon^{-3} \lambda_n \tilde{P}_n(x) \quad (4.6)$$

For monomials x^n , from (3.10) and (4.2)-(4.5) we have in the limit $\epsilon \rightarrow 0$,

$$\begin{aligned} L_0 x^n = \lim_{\epsilon \rightarrow 0} \frac{L_\epsilon}{\epsilon^3} x^n = & [-8Mn(n+2)(n+1+\beta) + 8n(\beta+1)(\beta+3) \\ & + \theta_n(16Mn^3 + (24\beta M + 48M)n^2 + (32M - 16\beta^2 + 8\beta^2 M - 48 + 48\beta M - 64\beta)n \\ & - 48\beta^2 - 8\beta^3 + 16\beta M + 8\beta^2 M - 48 - 88\beta)] x^n \\ & + [8Mn^3 + (24M + 8\beta M)n^2 + (16M + 16\beta M - 8\beta^2 - 32\beta - 24)n \\ & + \theta_n(-16Mn^3 - (24M + 16\beta M)n^2 + (64\beta + 48 + 16\beta^2 - 16\beta M - 8M)n \\ & + 32\beta + 24 + 8\beta^2 + 8\beta M)] x^{n-1} \\ & + [8Mn^3 - 32Mn + \theta_n(-16Mn^3 - 8\beta M n^2 + 40Mn + 8\beta M)] x^{n-2} \\ & + [-8Mn^3 + 32Mn + \theta_n(16Mn^3 - 24Mn^2 - 40Mn + 24M)] x^{n-3} \end{aligned} \quad (4.7)$$

where $\theta_n = \frac{1-(-1)^n}{2}$. This allows one to present the operator $L_0 = \lim_{\epsilon \rightarrow 0} \frac{L_\epsilon}{\epsilon^3}$ in the form

$$\begin{aligned}
L_0 = & (-8M + 8Mx + 8Mx^2 - 8Mx^3) \partial_x^3 R \\
& + [-12M/x + 24M + 4\beta M + (36M + 8\beta M)x - (12\beta M + 48M)x^2] \partial_x^2 R \\
& + (12Mx + 4\beta Mx^2 - 12M/x - 4\beta M) \partial_x^2 + [(24M + 16\beta M - 8\beta^2 - 32\beta - 24) \\
& + 24M/x^2 + (4\beta - 12)M/x + (8\beta^2 - 36\beta M - 48M - 4\beta^2 M + 32\beta + 24)x] \partial_x R \\
& + [(4\beta^2 M + 12\beta M)x - (12 + 4\beta)M/x + 24M + 8\beta M] \partial_x \\
& + [12M/x^3 + 4\beta M/x^2 + (12 + 4\beta^2 + 4\beta M + 16\beta)/x \\
& + (8\beta M + 4\beta^2 M - 44\beta - 24 - 4\beta^3 - 24\beta^2)] (1 - R),
\end{aligned} \tag{4.8}$$

where R is the reflection operator $Rf(x) = f(-x)$. We thus have that the polynomials $\tilde{P}_n^{(-1)}(x)$ are classical and satisfy the eigenvalue equation

$$L_0 \tilde{P}_n^{(-1)}(x) = \tilde{\lambda}_n \tilde{P}_n^{(-1)}(x), \tag{4.9}$$

where

$$\tilde{\lambda}_n = \begin{cases} -8Mn(n+2)(n+1+\beta) + 8n(\beta+1)(\beta+3) & n \text{ even} \\ 8M(n+1)(n+\beta)(n+2+\beta) - 8(n+2+\beta)(\beta+1)(\beta+3) & n \text{ odd} \end{cases}. \tag{4.10}$$

The lower degree eigensolutions of (4.9) can be obtained directly as a check and as examples. We take

$$\tilde{\lambda}_1 = (\beta+1)(\beta+3)(16M - 8(\beta+3)), \tag{4.11}$$

and find the first order polynomial solution

$$\tilde{P}_1^{(-1)}(x) = x - 1 + \frac{2\beta - 1 - \alpha}{2\beta - 3 - \alpha}. \tag{4.12}$$

The second-order polynomial solution is

$$\tilde{\lambda}_2 = (\beta+3)(16\beta + 16 - 64M), \tag{4.13}$$

$$\tilde{P}_2^{(-1)}(x) = x^2 - \frac{2(4M - \beta - 1)}{(5 + \beta)(2M - \beta - 1)}x + \frac{2(\beta + 1)}{(5 + \beta)(2M - \beta - 1)}. \tag{4.14}$$

and the third-order polynomial solution

$$\tilde{\lambda}_3 = (3 + \beta)(5 + \beta)(32M - 8 - 8\beta), \tag{4.15}$$

$$\tilde{P}_3^{(-1)}(x) = x^3 - \frac{4(-2M + 1 + \beta)}{(7 + \beta)(-4M + \beta + 1)}x^2 - \frac{4}{7 + \beta}x + \frac{8(1 + \beta)}{(7 + \beta)(5 + \beta)(-4M + \beta + 1)}. \tag{4.16}$$

Other polynomial eigensolutions can be obtained in the $q = -1$ limit of (3.1). The weight function (3.4) has the following moments

$$\tilde{c}_n = k \left(\frac{(q^2; q)_n}{(bq^3; q)_n} - \frac{1 - q^2}{1 - bq^3} M \delta_{n,0} \right). \tag{4.17}$$

Using the parametrization (4.1) and taking the limit $\epsilon \rightarrow 0$, we can directly obtain from the above relation the moments corresponding to the polynomials $\tilde{P}_n^{(-1)}$.

$$\mu_{2n} = \mu_{2n-1} = k \left[\frac{(1)_n}{(\beta/2 + 3/2)_n} - \frac{2}{3 + \beta} M \delta_{n,0} \right], \quad n = 1, 2, \dots, \tag{4.18}$$

where $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$ is the ordinary Pochhammer symbol. It is then easily verified that

$$w(x) = \tilde{k} \left(|x|(1-x^2)^{(\beta-1)/2}(1+x) - \frac{4}{(1+\beta)(3+\beta)} M \delta(x) \right), \tag{4.19}$$

where

$$\tilde{k} = \frac{\Gamma(\beta/2 + 3/2)}{\Gamma(\beta/2 + 1/2)}k = \frac{\beta + 1}{2}k \quad (4.20)$$

is the orthogonality measure for these polynomials. Indeed we see that

$$\int_{-1}^1 w(x)x^n dx = \mu_n, \quad n = 0, 1, 2, \dots, \quad (4.21)$$

with μ_n given by (4.18).

In the following we determine the three term recurrence relation that the polynomials $\tilde{P}_n^{(-1)}(x)$ verify. It is already known that the little q -Jacobi polynomials satisfy the relation

$$P_{n+1} + b_n P_n + u_n P_{n-1} = x P_n, \quad (4.22)$$

and that the recurrence coefficients are defined by

$$u_n = A_{n-1}C_n, \quad b_n = A_n + C_n,$$

where A_n, C_n are given as

$$A_n = q^n \frac{(1 - aq^{n+1})(1 - abq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})}, \quad C_n = aq^n \frac{(1 - q^n)(1 - bq^n)}{(1 - abq^{2n+1})(1 - abq^{2n})}. \quad (4.23)$$

Under the Geronimus transformation

$$\tilde{P}_n(x) = P_n(x) - B_n P_{n-1}(x), \quad (4.24)$$

with $B_n = \frac{\Phi_n}{\Phi_{n-1}}$ and Φ_n defined by (2.8), these result the three term recurrence relation

$$\tilde{P}_{n+1} + \tilde{b}_n \tilde{P}_n + \tilde{u}_n \tilde{P}_{n-1} = x \tilde{P}_n, \quad (4.25)$$

with coefficients

$$\tilde{u}_1 = \frac{\phi_1}{\phi_0^2}, \quad \tilde{u}_n = \frac{u_{n-1}B_n}{B_{n-1}}, \quad n = 2, 3, \dots \quad (4.26)$$

$$\tilde{b}_0 = b_0 + \frac{\phi_1}{\phi_0}, \quad \tilde{b}_n = b_n + B_{n+1} - B_n, \quad n = 1, 2, \dots \quad (4.27)$$

When we set $a = q^2, q = -e^\epsilon, b = -e^{\beta\epsilon}$ and take the limit $\epsilon \rightarrow 0$, Eq. (4.25) reduces to

$$\tilde{P}_{n+1}^{(-1)} + \tilde{b}_n^{(-1)} \tilde{P}_n^{(-1)} + \tilde{u}_n^{(-1)} \tilde{P}_{n-1}^{(-1)} = x \tilde{P}_n^{(-1)}. \quad (4.28)$$

Here the coefficients are

$$\tilde{b}_n^{(-1)} = \lim_{\epsilon \rightarrow 0} (b_n + B_{n+1} - B_n) = b_n^{(-1)} + \lim_{\epsilon \rightarrow 0} (B_{n+1} - B_n) \quad (4.29)$$

$$\tilde{u}_n^{(-1)} = \lim_{\epsilon \rightarrow 0} \frac{u_{n-1}B_n}{B_{n-1}} = u_{n-1}^{(-1)} \lim_{\epsilon \rightarrow 0} \frac{B_n}{B_{n-1}}, \quad (4.30)$$

where

$$u_n^{(-1)} = -\frac{n(n+2)}{(2n+1+\beta)(2n+3+\beta)}, \quad b_n^{(-1)} = 1 \quad (4.31)$$

when n is even, and

$$u_n^{(-1)} = -\frac{(n+\beta)(n+2+\beta)}{(2n+1+\beta)(2n+3+\beta)}, \quad b_n^{(-1)} = -1 \quad (4.32)$$

when n is odd. Note that

$$\lim_{\epsilon \rightarrow 0} B_n = \lim_{q \rightarrow -1} \frac{\Phi_n}{\Phi_{n-1}} = \begin{cases} \frac{n+2}{2n+1+\beta} \frac{M - [(3+\beta)(1+\beta)]/[(n+2)(n+1+\beta)]}{M - [(3+\beta)(1+\beta)]/[n(n+1+\beta)]} & \text{n even} \\ -\frac{n+2+\beta}{2n+1+\beta} \frac{M - [(3+\beta)(1+\beta)]/[(n+1)(n+2+\beta)]}{M - [(3+\beta)(1+\beta)]/[(n+1)(n+\beta)]} & \text{n odd} \end{cases} \quad (4.33)$$

As a final observation, let us identify the matrix orthogonal polynomials that the even (or odd) part of the $\tilde{P}_n^{(-1)}(x)$ define. Split the polynomials $\tilde{P}_n^{(-1)}(x)$ into its even (E_n) and odd (O_n) parts:

$$\tilde{P}_n^{(-1)}(x) = E_n(x) + O_n(x). \quad (4.34)$$

From the recurrence relation (4.28), we have

$$\begin{aligned} x^2 E_n = E_{n+2} &+ (\tilde{b}_{n+1}^{(-1)} + \tilde{b}_n^{(-1)})E_{n+1} + (\tilde{u}_{n+1}^{(-1)} + \tilde{u}_n^{(-1)} + (\tilde{b}_n^{(-1)})^2)E_n \\ &+ (\tilde{b}_n^{(-1)} + \tilde{b}_{n-1}^{(-1)})\tilde{u}_n^{(-1)}E_{n-1} + \tilde{u}_{n-1}^{(-1)}\tilde{u}_n^{(-1)}E_{n-2}. \end{aligned} \quad (4.35)$$

With the redefinition $E_n = \sigma_n F_n$, $\sigma_n = \sqrt{\tilde{u}_1^{(-1)}\tilde{u}_2^{(-1)}\cdots\tilde{u}_n^{(-1)}}$, it is easy to see that the polynomial F_n satisfies the five-term recurrence relation,

$$x^2 F_n(x) = c_{n,0}F_n + c_{n,1}F_{n-1} + c_{n+1,1}F_{n+1} + c_{n,2}F_{n-2} + c_{n+2,2}F_{n+2} \quad (4.36)$$

where the coefficients are

$$c_{n,0} = (\tilde{u}_{n+1}^{(-1)} + \tilde{u}_n^{(-1)} + (\tilde{b}_n^{(-1)})^2), c_{n,1} = (\tilde{b}_{n-1}^{(-1)} + \tilde{b}_n^{(-1)})\sqrt{\tilde{u}_n^{(-1)}}, c_{n,2} = \sqrt{\tilde{u}_n^{(-1)}\tilde{u}_{n-1}^{(-1)}}. \quad (4.37)$$

From the theorem in [27], the matrix polynomials $\{P_n(x)\}$ defined by

$$P_n(x) = \begin{pmatrix} R_{2,0}(F_{2n})(x) & R_{2,1}(F_{2n})(x) \\ R_{2,0}(F_{2n+1})(x) & R_{2,1}(F_{2n+1})(x) \end{pmatrix} \quad (4.38)$$

satisfy the matrix three term recurrence relation

$$xP_n(x) = D_{n+1}P_{n+1}(x) + E_nP_n(x) + D_n^*P_{n-1}(x) \quad (4.39)$$

where

$$D_n = \begin{pmatrix} c_{2n,2} & 0 \\ c_{2n,1} & c_{2n+1,2} \end{pmatrix}, \quad E_n = \begin{pmatrix} c_{2n,0} & c_{2n+1,1} \\ c_{2n+1,1} & c_{2n+1,0} \end{pmatrix}. \quad (4.40)$$

The polynomials $R_{N,m}(p)(x)$ are defined by

$$R_{N,m}(p)(x) = \sum_n \frac{p^{(nN+m)}(0)}{(nN+m)!} x^n. \quad (4.41)$$

5 Conclusion

To sum up, we have added to the exploration of -1 polynomials by introducing some -1 Krall-Jacobi polynomials. We focused on the simplest positive definite case. They have been obtained from generalized q -Jacobi polynomials through a limiting procedure and have remarkable features. Noteworthy is the fact that they obey a third-order differential-difference eigenvalue equation involving the reflection operator. The 3-term recurrence relation has also been determined. Finally let us stress that this present an interesting example of OPs whose measure involves a discrete mass at the center of the orthogonality interval (rather than at its boundary which is more common).

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